Interpretation of the conductivity of a Luttinger liquid

N. M. Vildanov*

I. E. Tamm Department of Theoretical Physics, P. N. Lebedev Physics Institute, 119991 Moscow, Russia

(Received 24 January 2010; published 2 July 2010)

An interpretation of the conductivity of a Luttinger liquid in terms of electron-hole pair creation in the corresponding one-dimensional electron gas with Dirac Hamiltonian is given. Full counting statistics of electron-hole pair creation in arbitrary external electric field is considered.

DOI: 10.1103/PhysRevB.82.033101 PACS number(s): 71.10.Pm, 23.20.Ra, 72.15.Nj

There is vast literature on Luttinger liquids (see, e.g., Refs. 1–5). Their various properties has been thoroughly studied since the discovery of bosonization by Tomonaga,[6] which was further developed by Luttinger[7] and Haldane.[8] Here we readdress a question concerning the response of a pure Luttinger liquid to an arbitrary external electric field. We give a simple picture of transport in Luttinger liquids based on the electron-hole pair creation in the corresponding one-dimensional electron gas with Dirac Hamiltonian. Although the nonlocal ac conductivity of a Luttinger liquid $\sigma(x,\omega)=\frac{e^2}{h}\cos\frac{\omega x}{v_F}$ has been known long ago and studied in detail,[4] it seems that the formula (9) has not been fully appreciated in the existing literature. In this paper we want to emphasis that it has a simple meaning. The concept of Luttinger liquid is one of the most important concepts in modern condensed matter physics and the lack of such simple interpretation of the formula (9) should be addressed.

The formula (9) can be obtained straightforwardly from the ac conductivity of Luttinger liquid $\sigma(x,\omega)$, but it is instructive to present here the whole derivation, first, to set notations and second, because we will make use of some results in the subsequent discussions (in the context of full counting statistics).

We begin by describing the standard technique of evaluation of the response function of Luttinger liquid. The action for Luttinger liquid interacting with an electromagnetic field has the form[4]

$$S=\frac{1}{2}\int dxdt \times\left[v_F(\partial_x\varphi)^2-\frac{1}{v_F}(\partial_t\varphi)^2+\frac{2e}{\sqrt{\pi}}(A_0\partial_x\varphi-A_x\partial_t\varphi)\right], \quad (1)$$

where A_μ is the vector potential of the electromagnetic field, and

$$\rho=e\partial_x\varphi/\sqrt{\pi}, \quad (2)$$

$$j=-e\partial_t\varphi/\sqrt{\pi} \quad (3)$$

are, respectively, charge and current densities. Variating in φ, one obtains equations of motion

$$v_F\partial_x^2\varphi-\frac{1}{v_F}\partial_t^2\varphi=\frac{e}{\sqrt{\pi}}E(x,t), \quad (4)$$

where $E(x,t)=-\partial_xA_0+\partial_tA_x$ is the strength of the electric field. Performing the Fourier transform of Eq. (4) gives

$$v_F\partial_x^2\bar{\varphi}+\frac{1}{v_F}\omega^2\bar{\varphi}=\frac{e}{\sqrt{\pi}}E(x,\omega). \quad (5)$$

The general solution to this equation is

$$\bar{\varphi}(x,\omega)=Ae^{i(\omega/v_F)x}+Be^{-i(\omega/v_F)x} +\frac{e}{\sqrt{\pi}\omega}\int_{-\infty}^{x}E(y,\omega)\sin\left[\frac{\omega}{v_F}(x-y)\right]dy, \quad (6)$$

where A and B are arbitrary constants. From the requirement that at $x\to\pm\infty$ only $e^{\pm i(\omega/v_F)x}$ harmonics survive we find

$$\bar{\varphi}(x,\omega)=-\frac{ie}{2\omega\sqrt{\pi}}\int_{-\infty}^{\infty}E(y,\omega)e^{i(\omega/v_F)(y-x)}dy +\frac{e}{\omega\sqrt{\pi}}\int_{-\infty}^{x}E(y,\omega)\sin\left[\frac{\omega}{v_F}(x-y)\right]dy. \quad (7)$$

The Fourier transform of the current density $j(x,t)=-e\partial_t\varphi(x,t)/\sqrt{\pi}$ is given by

$$j(x,\omega)=\frac{ie\omega}{\sqrt{\pi}}\bar{\varphi}(x,\omega). \quad (8)$$

It is now simple to derive the result

$$j(x,t)=\frac{e^2}{h}\int_{-\infty}^{\infty}E(y,t-|x-y|/v_F)dy. \quad (9)$$

We assume for brevity that $\hbar=1$, i.e., $2\pi=h$.

Luttinger liquid emerges when one bosonizes the Hamiltonian of one-dimensional massless Dirac fermions. The Lagrangian is[2]

$$L=\int dx\bar{\psi}\gamma^\mu(\mathrm{i}\partial_\mu+eA_\mu)\psi, \quad (10)$$

where $\bar{\psi}=\psi^\dagger\gamma^0$, γ^μ are matrices satisfying the condition $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}$, $g^{\mu\nu}$ is the metric tensor $g^{00}=0$, $g^{11}=-1$. We have put $v_F=1$ for simplicity.

It is possible to integrate out fermions exactly in this model using the following correspondence between fermionic current operators $J^\mu=\bar{\psi}\gamma^\mu\gamma$ and the bosonic field φ

$$J^{\mu} = \frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial_{\nu}\varphi \tag{11}$$

with the action

$$S = \int d^2x(\nabla\varphi)^2, \tag{12}$$

which is valid in one dimension only.[9] The result for the effective action of the electromagnetic field is[10]

$$S_{eff}[A] = \frac{1}{2\pi}\int d^2x d^2y F_{\mu\nu}(x)D(x,y)F^{\mu\nu}(y), \tag{13}$$

in which D is the outgoing-wave Green's function defined by

$$\partial^2 D(x,x') = \delta(x-x'). \tag{14}$$

In Fourier space it is given by

$$D(\omega,q) = \frac{1}{\omega^2 - q^2 + i0 \text{ sign } \omega}. \tag{15}$$

The term $+i0$ sign ω ensures the causality.

It is possible to obtain the current induced by electromagnetic field by variating this expression in A^{μ}. The answer is[10]

$$j_{\mu}(x) = -\frac{e^2}{\pi}A_{\mu}(x) - \frac{e^2}{\pi}\partial_{\mu}\int dx' D(x,x')\partial'_{\nu}A^{\nu}(x'). \tag{16}$$

This expression was first obtained by Schwinger almost half century ago.[10] He developed the theory of one-dimensional electrodynamics and showed that this model is exactly solvable. We will assume that the field A_{μ} entering Eq. (16) is the external electric field.

Now we need to simplify Eq. (16). We will assume for simplicity that $A_x=0$. Then substituting Eq. (15) in Eq. (16), we obtain after partial integration

$$j(x,t) = \frac{e^2}{\pi}\int dy d\tau E(y,\tau)\frac{\partial}{\partial t}\int \frac{d\omega}{2\pi}\frac{dq}{2\pi}\frac{e^{iq(x-y)-i\omega(t-\tau)}}{\omega^2 - q^2 + i0 \text{ sign } \omega}. \tag{17}$$

ω-integration gives

$$j(x,t) = \frac{e^2}{2\pi}\int dy \int_{-\infty}^{t} d\tau E(y,\tau) \times \int \frac{dq}{2\pi}e^{iq(x-y)}(e^{-i|q|(t-\tau)} + e^{i|q|(t-\tau)}). \tag{18}$$

Simple calculations yield the final expression which coincides with Eq. (9), thus providing another derivation of Eq. (9).

The aim we followed when deriving Eq. (9) by different methods was not to show that both methods lead to the same result, but rather to show that in combined fermion-boson language the transport in both systems becomes quite transparent.

Equation (9) has a simple explanation. We recall here the fact that external electric field creates electron-hole pairs. The rate of pair creation per unit length equals[11–17] (for massless Dirac fermions)

$$\Gamma = \frac{e|E|}{2\pi}. \tag{19}$$

Although in Refs. 11–17 this expression was obtained only for a uniform, constant in time electric field, it is reasonable to assume that it is correct for arbitrary electric fields. The fact that in one dimension the pair creation rate in the system of massless Dirac fermions is local drastically simplifies all the situation. After the pair is created, the electron will move in one direction with the velocity v_F, whereas the hole will move in the opposite direction with the same velocity. If the pair is created at the point y it takes the electron or the hole the time $|x-y|/v_F$ to reach the point x (whether it is the electron or the hole reaches the point x, obviously depends on the direction of the electric field at the point y at the moment $t-|x-y|/v_F$). Then one needs to integrate over the whole space taking into account this fact. Hence the retarded structure in Eq. (9). Let us stress here that in this picture, the effect of the external electric field is the creation of new electron-hole pairs only. The created pairs move as free particles. This can be seen from Eq. (4) since $f_1(x-v_Ft)+f_2(x+v_Ft)$ is always a solution of the homogenous equation. This reasoning can also serve as an elementary derivation of the formula (9).

Let us now discuss what changes in the above picture when the Luttinger liquid parameter K does not equal to 1 (as discussed above, $K=1$ corresponds to interactionless electron system; in the system of electrons with short range interaction the parameter is $K<1$ for repulsive and $K>1$ for attractive interactions, respectively). In this case the charge and the Fermi velocity are renormalized according to

$$u = \frac{v_F}{K}, \quad e^* = e\sqrt{K}. \tag{20}$$

Then Eq. (9) and the above picture remain valid in the general case provided that e and v_F are replaced by Eq. (20), $E(x,t)$ being the external field, but not the actual electric field. The setup relevant for experiments is a Luttinger liquid coupled to leads.[18] Potential drop between the leads is renormalized because of interactions.[19,20] It is due to this fact that the dc conductivity is universal (interaction independent)[19–24] and the ac conductivity is not given by $\sigma(x,\omega)=\frac{e^2}{h}\cos\frac{\omega x}{v_F}$ with renormalized e and v_F, but by a rather more complicated expression.[25]

Now we turn to statistics of electron-hole pair creation (it will suggest us that the above picture is valid not only on the average, but also when characterized by higher order moments). It is sufficient to find the characteristic function of the corresponding distribution which is defined as

$$\Lambda(\chi) = \langle e^{i\chi(\hat{Q}/e)}\rangle, \tag{21}$$

where

$$\hat{Q} = -\frac{e}{\sqrt{\pi}}\int_{-\infty}^{\infty}\partial_t\varphi(x,t)dt \tag{22}$$

is the charge transferred through the point x. As we will see, the form in which Eq. (22) is written is very convenient for calculations. After averaging $\exp(i\chi\hat{Q}/e)$, using the action for the boson field φ Eq. (1) and subtracting the equilibrium contribution to charge fluctuations, we obtain

$$\Lambda(\chi) = \exp\left[\frac{ie\chi}{\pi}\int dy d\tau E(y,\tau)\int_{-\infty}^{\infty}\partial_t D(t-\tau,x-y)dt\right] \tag{23}$$

The derivative in Eq. (23) was calculated earlier [see Eqs. (17) and (18)] and is given by a sum of delta functions

$$\frac{1}{2}\theta(t-\tau)[\delta(t-\tau-x+y)+\delta(t-\tau+x-y)].$$

Finally we obtain

$$\Lambda(\chi) = e^{(ie\chi/2\pi)\int E(y,\tau)dyd\tau}. \tag{24}$$

Introducing the following notation for the total charge transferred

$$N_{\mathrm{a}} = \frac{e}{2\pi}\int E(y,\tau)dyd\tau, \tag{25}$$

this can be rewritten as

$$\Lambda(\chi) = e^{i\chi N_{\mathrm{a}}}. \tag{26}$$

We see that charge does not fluctuate. This is expected and for the case of uniform electric field can be derived also by Landauer formalism. Consider a setup, consisting of two electrodes at $x<0$ and $x>L$ with a linear potential barrier formed between them by electric field E (see, e.g., Ref. 13). The transmission probability of Dirac electrons with mass m through this barrier is

$$T = e^{-\pi m^2/eE}. \tag{27}$$

The characteristic function expressed in terms of T reads[26]

$$\Lambda_1(\chi) = [1+(e^{i\chi}-1)T]^{N_1}, \tag{28}$$

where N_1 is the so-called number of attempts to traverse the barrier during the time interval Δt

$$N_1 = \frac{eEL\Delta t}{2\pi}.$$

This agrees with Eqs. (25) and (26) for $m=0$. The average and the mean square deviation of the transferred charge are given by

$$\langle N\rangle = N_1 T, \quad \langle(\delta N)^2\rangle = N_1 T(1-T). \tag{29}$$

We note here, that the formula (28) for the full counting statistics of pair creation in the uniform electric field can be easily generalized to higher dimensions. The derivation of the full counting statistics by field theoretical methods, if it exists, would be very complicated. However, the Landauer formalism together with the Levitov formula[26] provides a very simple derivation. For $D=2$ by direct application of the results of[26] and taking into account that the momentum dependent transmission probabilities are given by Eq. (27) the mass m being substituted by $\sqrt{m^2+k^2}$ (k–transverse momenta), we have

$$\ln \Lambda_2(\chi) = N_1 L_\perp \int \frac{dk}{2\pi}\ln[1+(e^{i\chi}-1)e^{-\pi[(m^2+k^2)/eE]}]$$
$$= -N_2\sum_{n=1}^{\infty}\frac{(1-e^{i\chi})^n}{n^{3/2}}T^n,$$

where

$$N_2 = \frac{(eE)^{3/2}S\Delta t}{(2\pi)^2}, \quad S = LL_\perp$$

and T is still given by Eq. (27). The average and the mean square deviation are (see also,[12] where the Fano factor $F=1-\frac{1}{2^{1/2}}$ of a graphene p-n junction was obtained)

$$\langle N\rangle = N_2 T, \quad \langle(\delta N)^2\rangle = N_2 T\left(1-\frac{T}{2^{1/2}}\right).$$

In $D=3$ one has

$$\ln \Lambda_3(\chi) = N_1 S_\perp \int \frac{d^2k}{(2\pi)^2}\ln[1+(e^{i\chi}-1)e^{-\pi[(m^2+k^2)/eE]}]$$
$$= -N_3\sum_{n=1}^{\infty}\frac{(1-e^{i\chi})^n}{n^2}T^n, \tag{30}$$

where

$$N_3 = \frac{(eE)^2 V\Delta t}{(2\pi)^3}, \quad V = LS_\perp.$$

The average and the mean square deviation:

$$\langle N\rangle = N_3 T, \quad \langle(\delta N)^2\rangle = N_3 T\left(1-\frac{T}{2}\right).$$

$\langle N\rangle$ is in agreement with the results known in quantum electrodynamics (see Refs. 16 and 17 and references therein).

In summary, we have shown that the well known formula (9), which is just a Fourier transform of the response function given in textbooks, has a simple interpretation as creation of electron-hole pairs with the rate $\frac{e|E|}{h}$ and their subsequent motion with the velocity v_F. This can serve also as an elementary derivation of Eq. (9), provided that one knows the dc conductivity $\frac{e^2}{h}$ of Luttinger liquid and the fact that pair creation in the system of one-dimensional massless fermions is local. This constitutes the main result of the paper. In conclusion we considered the full counting statistics of pair creation in arbitrary electric field which turns out to be trivial (there is no fluctuations) as is expected in analogy with the uniform electric field case.

The author would like to acknowledge S. M. Apenko and P. I. Arseev for valuable discussions of the results.

*niyazvil@list.ru

[1] J. Voit, Rep. Prog. Phys. **57**, 977 (1994).
[2] A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, Cambridge, England, 1998).
[3] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, Oxford, 2004).
[4] D. L. Maslov, in *Nanophysics: Coherence and Transport*, edited by H. Bouchiat, Y. Gefen, S. Guéron, G. Montambaux, and J. Dalibard (Elsevier, Amsterdam, 2005).
[5] A. Kawabata, Rep. Prog. Phys. **70**, 219 (2007).
[6] S. Tomonaga, Prog. Theor. Phys. **5**, 544 (1950).
[7] J. M. Luttinger, J. Math. Phys. **4**, 1154 (1963).
[8] F. D. M. Haldane, J. Phys. C **14**, 2585 (1981).
[9] S. Coleman, Phys. Rev. D **11**, 2088 (1975).
[10] J. S. Schwinger, Phys. Rev. **128**, 2425 (1962).
[11] J. S. Schwinger, Phys. Rev. **82**, 664 (1951).
[12] V. V. Cheianov and V. I. Falko, Phys. Rev. B **74**, 041403 (2006).
[13] A. V. Andreev, Phys. Rev. Lett. **99**, 247204 (2007).
[14] A. V. Shytov, Nan Gu, and L. S. Levitov, arXiv:0708.3081 (unpublished).
[15] D. Allor, T. D. Cohen, and D. A. McGady, Phys. Rev. D **78**, 096009 (2008).
[16] T. D. Cohen and D. A. McGady, Phys. Rev. D **78**, 036008 (2008).
[17] E. S. Fradkin, D. M. Gitman, and S. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer-Verlag, Berlin, 1991).
[18] C. L. Kane and M. P. A. Fisher, Phys. Rev. Lett. **68**, 1220 (1992); Phys. Rev. B **46**, 15233 (1992).
[19] A. Kawabata, J. Phys. Soc. Jpn. **65**, 30 (1996).
[20] Y. Oreg and A. M. Finkel'stein, Phys. Rev. B **54**, R14265 (1996).
[21] I. Safi and H. J. Schulz, Phys. Rev. B **52**, R17040 (1995).
[22] D. L. Maslov and M. Stone, Phys. Rev. B **52**, R5539 (1995).
[23] V. V. Ponomarenko, Phys. Rev. B **52**, R8666 (1995).
[24] A. Yu. Alekseev, V. V. Cheianov, and J. Fröhlich, Phys. Rev. Lett. **81**, 3503 (1998).
[25] Ya. M. Blanter, F. W. J. Hekking, and M. Buttiker, Phys. Rev. Lett. **81**, 1925 (1998), and references therein.
[26] L. S. Levitov, H. W. Lee, and G. B. Lesovik, J. Math. Phys. **37**, 4845 (1996).